MA 242 : Partial Differential Equations (August-December, 2018)

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Problem set 3

1. Let $u, v \in C^2(\overline{\Omega})$. Using the Gauss-Green theorem, the following identites

   (a) $\displaystyle\int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}.$

   (b) $\displaystyle\int_\Omega v\Delta u = -\int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v.$

   (c) $\displaystyle\int_\Omega (v\Delta u - u\Delta) = \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu} v - u\frac{\partial v}{\partial \nu} \right).$

   Here, $\dfrac{\partial u}{\partial \nu} = \nabla u \cdot \nu$ is the normal derivative and $\nabla = \left(\dfrac{\partial}{\partial x_1}, \cdots \dfrac{\partial}{\partial x_n} \right)$ is the *grad* operator.

2. When $n = 2$, derive the Laplace operator Δ in polar form.

3. [Spherical Symmetry] Let R is a rotational matrix, that is $RR^t = I$ and u be harmonic in $\mathbb{R}^n$. Define, v by $v(x) = u(Rx)$, then show that v is harmonic in $\mathbb{R}^n$.

4. Let $v(r) = u(|x|)$ where $r = |x| = (\sum x_i^2)^{1/2}$. Show that

   $$\Delta u \equiv \ddot{v}(r) + \frac{n-1}{r}\dot{v}.$$

   Solve the equation to obtain the fundamental solution ϕ.

5. Let ϕ be the fundamental solution of the Laplace operator. Show that there exists a constant $C > 0$ such that

   $$|D\phi(x)| \leq \frac{C}{|x|^{n-1}}, \; |D^2\phi(x)| \leq \frac{C}{|x|^n}, \quad x \neq 0.$$

6. Let ϕ be the fundamental solution for the Laplace operator Δ. Prove that ϕ and $\dfrac{\partial \phi}{\partial x_i}$ for all $i = 1, \cdots n$ are locally integrable, but $\dfrac{\partial^2 \phi}{\partial x_i \partial x_j}$ for all $i, j = 1, \cdots n$ are not locally integrable.

7. Let $f \in C_c^2(\mathbb{R}^n)$ and ϕ be the fundamental solution for the Laplace operator Δ. Define $I_\varepsilon = \displaystyle\int_{B(0,\varepsilon)} \phi(y)(\Delta f)(x-y)dy$. Show that there exists a constant $C > 0$ such that

$$|I_\varepsilon| \leq \begin{cases} C\varepsilon^2 |log \varepsilon|, \text{ if } n = 2 \\ C\varepsilon^2 \text{ if } n \geq 3 \end{cases}$$

Also compute $\dfrac{\partial \phi}{\partial \nu}$ on $\partial B(0, \varepsilon)$.

8. Let Ω be a domain in $\mathbb{R}^2$ symmetric about the x-axis and let $\Omega^+ = \{(x,y) : y > 0\}$ be upper part. Assume $u \in C(\overline{\Omega^+})$ is harmonic in Ω^+ with $u = 0$ on $\partial\Omega^+ \cap \{y = 0\}$. Define

$$v(x,y) = \begin{cases} u(x,y) & if \ y \geq 0, \ (x,y) \in \Omega, \\ -u(x,-y) & if \ y < 0, \ (x,y) \in \Omega. \end{cases}$$

Show that v is harmonic.

9. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of

$$\Delta u + \sum a_k(x) \frac{\partial u}{\partial x_k} + c(x)u = 0 \text{ in } \Omega$$

with $c(x) < 0$ in Ω, $u = 0$ on $\partial\Omega$ and a_k's are smooth. Show that $u \equiv 0$.

10. Consider the PDE, $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$ where λ is a scalar and Ω is a bounded open set. If $\lambda \leq 0$, prove that $u \equiv 0$ and there is no non-trivial solution.

11. Let $u \in C^2(\overline{B(0;1)})$ solves $-\Delta u = f$ in $B(0;1)$, $u = 0$ on $\partial B(0;1)$. Show that there exists $C > 0$ such that

$$\max_{x \in B(0;1)} |u(x)| \leq C \max_{x \in B(0;1)} |f|.$$

(Hint: Consider the problem with $f = 1$ and $f = M$ where $M = \max_{x \in B(0;1)} |f|$.) More generally, if u solves $-\Delta u = f$ in $B(0;1)$, $u = g$ on $\partial B(0;1)$, then

$$\max_{x \in B(0;1)} |u(x)| \leq C \left(\max_{x \in \partial B(0;1)} |g| + \max_{x \in B(0;1)} |f| \right).$$

12. Show that the mapping $x \to \dfrac{x - e_n}{|x + e_n|}$, $e_n = (0, \ldots 1)$, is a C^∞ function mapping the upper half space $\mathbb{R}^n_+$ onto the unit ball $B_1(0)$ in a one-one fashion. Further, show that the boundary $\{x \in \mathbb{R}^n : x_n = 0\}$ is mapped onto the unit sphere $S_1(0) = \partial B_1(0)$.

13. Use the mapping in the above Exercise 12 and Poisson formula for $\mathbb{R}^n_+$ to derive Poisson formula for $B_1(0)$.

14. Let Ω be an open, bounded set in $\mathbb{R}^N$. Suppose $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $\Delta u = -1$ in Ω, $u = 0$ on $\partial\Omega$. Show that for $x \in \Omega$, $u(x) \geq \frac{1}{2N}(d(x, \partial\Omega))^2$. (Suggestion: For fixed $x_0 \in \Omega$, consider the harmonic function $u(x) + \frac{1}{2N}|x - x_0|^2$, $x \in \Omega$.)

15. If u is a harmonic function in $\mathbb{R}^n$ satisfying $|u(x)| \leq C(1+|x|^k)$, for some non-negative integer k and all $x \in \mathbb{R}^n$, show that u is a polynomial of degree at most k. In fact, the result is true for any non-negative real k, then u is a polynomial of degree at most $[k]$.

16. **(Harnack's inequality)** Let $u \geq 0$ be harmonic in a domain Ω. Let $V \subset\subset \Omega$ be connected, open and let $d = d(V, \partial\Omega)$ be the distance from V to the boudnary $\partial\Omega$. Use MVT in suitable open balls to prove that
$$2^n u(y) \geq u(x) \geq 2^{-n} u(y)$$
for all $x, y \in V$ satisfying $|x - y| \leq \frac{r}{4}$. Use this estimate to prove the following: There are constants $C_1, C_2 > 0$ depending on V such that
$$C_1 u(y) \geq u(x) \geq C_2 u(y)$$
for all $x, y \in V$.